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A Locally Most Powerful Rank Test For the Location Parameter Of a Double Exponential Distribution

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PREPARED UNDER
CONTRACT NO. NONR-285(38)
OFFICE OF NAVAL RESEARCH

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A Locally Most Powerful Rank Test
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A Locally Most Powerful Rank Test For the Location Parameter Of a Double Exponential Distribution

1. SUMMARY. The locally most powerful rank test (L.M.P.R.T) for the location parameter of the double exponential c.d.f.

$$L(x,\theta) = \begin{cases} \frac{1}{2}e^{(x-\theta)} & x \leq \theta \\ 1-\frac{1}{2}e^{(\theta-x)} & x \geq \theta \end{cases}$$

is examined. This test was originally found by Birnbaum [2] whose unpublished derivation is given for completeness in Section 2. A remark of Ansari and Bradley [1], which asserts that the Chernoff-Savage theorem [5] holds under certain weaker conditions is used to prove the asymptotic normality of Birnbaum's test.[*] Comparing this test with the likelihood ratio test and making use of Pittman's definition of asymptotic relative efficiency (A.R.E.) we find that Birnbaum's test is asymptotically efficient. The A.R.E. of the latter to the likelihood ratio test is one for symmetric distribution and otherwise is shown to vary between zero and infinity. The latter result is interesting in contrast with the situation for the normal distribution for which the A.R.E. of the L.M.P.R.T. (the Fisher-Yates) to the likelihood ratio test (the "t" test) is always greater than or equal to one.

2. Introduction - The Birnbaum Test

Let $x_1, x_2, \ldots, x_{n_1}$, $y_1, y_2, \ldots, y_{n_2}$ be $N = n_1 + n_2$ independent random variables, the x_i's having unknown absolutely continuous c.d.f. P_X and the y_i's having unknown absolutely continuous c.d.f.

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[*] Alternatively, a theorem of Hájek under slightly different assumptions proves the same result.

P_Y. We consider the problem of testing

$$H_0 : P_X = L(x,\phi_0),\ P_Y = L(y,\phi_0)$$

against

$$H_1 : P_X = L(x,\theta),\ P_Y = L(y,\phi),$$

where $-\infty < \phi < \phi_0 < \theta < \infty$ and we use only the ranks.

Let

$$z_{Nj} = \begin{cases} 1 & \text{if the } j^{th} \text{ smallest of the pooled sample of } N \text{ observations is an } x \\ 0 & \text{otherwise .} \end{cases}$$

Capon [4] has shown that, under regularity conditions which are met here, the L.M.P.R.T. is of the form

$$\phi(x_1,\dots,x_n,y_1,\dots,y_n) = \begin{cases} 0 & \text{if } B(z_{N1},\dots,z_{NN}) < C \\ 1 & \text{if } B(z_{N1},\dots,z_{NN}) > C \\ k_a & \text{if } B(z_{N1},\dots,z_{NN}) = C \end{cases}$$

where

$$B_N = B_N(z) = B(z_{N1},\dots,z_{NN}) = \frac{1}{n_1}\sum_{i=1}^{N} a_{Ni}z_{Ni}$$

$$a_{Ni} = E_{\phi_0\phi_0}\left(\frac{\partial}{\partial\theta} \log \ell\,(w_i\theta) \Big|_{\theta = \phi_0} \right)$$

$$\ell\,(x,\theta) = L'(x,\theta) = \frac{1}{2}\,e^{-|x-\theta|} \qquad -\infty < x < \infty$$

and w_i is the i^{th} smallest of the N observations. The notation $E_{\theta,\phi}$ indicates expectation is taken under the assumption that the x's and y's are distributed according to $\ell\,(x,\theta)$ and $\ell(y,\theta)$ respectively.

We find

$$E_{\phi_0\phi_0}\left(\frac{\partial}{\partial\theta}\ \log\ \mathcal{L}\ (w_i\theta)\Big|_{\phi_0}\right) = E_{\phi_0\phi_0}\{(S_{\phi_0}(w_i))\}$$

where

$$S_{\phi_0}(w) = \begin{cases} -1 & w < \phi_0 \\ 1 & w > \phi_0 \end{cases} .$$

Thus

$$\begin{aligned} a_{Ni} &= P\ \{i \text{ or more } x_i > \phi_0\} - P\ \{N-i \text{ or less } x_i < \phi_0\} \\ &= \sum_{j=i}^{N} \binom{N}{j} \left(\frac{1}{2}\right)^N - \sum_{j=0}^{i-1} \binom{N}{j} \left(\frac{1}{2}\right)^N \\ &= \left(\frac{1}{2}\right)^{N-1} \sum_{j=i}^{N} \binom{N}{j} - 1 . \end{aligned}$$

Letting $B_N = \frac{1}{n_1} \sum a_{Ni} z_{Ni}$, a statistic originally described in [2], we see that the Birnbaum test, a linear combination of binomial coefficients, is the L.M.P.R.T. for the location parameter of the exponential.

3. Asymptotic Normality of B_N

Capon [4] has obtained from Theorem 1 of Chernoff and Savage [5] and a simple extension of their Theorem 2, that a L.M.P.R.T. has, asymptotically, a normal distribution if a certain smoothness condition is satisfied. In our case the condition is

$$\left| J_L^{(i)}(H) \right| = \left| \frac{d^{(i)} J_L(H)}{dH^{(i)}} \right| \leq K(H(1-H))^{-i-\frac{1}{2}+\delta}$$

for $i = 0,1,2$ and for some $\delta > 0$ where k is a constant and

where

$$0 < \lim_{N \to \infty} \frac{n_1}{n_2} = r < \infty , \quad J_L(H) = \begin{cases} -1 & H < \frac{1}{2} \\ 1 & H > \frac{1}{2} \end{cases} .$$

This is clearly satisfied for all H except for the point $H = \frac{1}{2}$ where the derivative does not exist. However, Ansari and Bradley [1] have remarked that the conclusion of the Chernoff-Savage theorem holds even with such a discontinuity. They assert that this follows immediately upon inspection of the proof. (the present author has not checked this).[**] From this we conclude that for absolutely continuous $G_\theta(x)$,

$$\lim \text{Prob}_G \left\{ \frac{B_N - E_{\theta\phi}(B_N)}{\sigma_{\theta\phi}(T_N)} \leq t \right\} = \int_{-\infty}^{t} (2\pi)^{-\frac{1}{2}} e^{-\frac{x^2}{2}} dx$$

where

$$E_{\theta\phi}(B_N) = \int_{-\infty}^{\infty} J_L[\frac{n_1}{N} G_\phi(x) + \frac{n_2}{N} G_\theta(x)] dG_\theta(x)$$

$$= 1 - 2G_\theta(x^*)$$

and where x^* is that x for which $\frac{n_1}{N} G_\phi(x) + \frac{n_2}{N} G_\theta(x) = \frac{1}{2}$.

Also

$$N\sigma^2_{\theta\phi}(B_N) = \frac{2n_2}{N} \left\{ \iint_{-\infty < x < y < \infty} G_\phi(x)(1-G_\phi(y)) J_L'(H_{\theta\phi}(x) J_L'(H_{\theta\phi}(y)) dG_\theta(x) dG_\theta(y) \right.$$

$$\left. + \frac{n_2}{n_1} \iint_{-\infty < x < y < \infty} G_\theta(x)(1-G_\theta(y)) J_L'(H_{\theta\phi}(x) J_L'(H_{\theta\phi}(y)) dG_\phi(x) dG_\phi(y) \right\} ,$$

[**] In a recent article, Jaroslav Hajek has pointed out that asymptotic normality can be proved under other regularity conditions which we may add here without substantially changing any of the results. (Dec. 1962 Annals of Math. Stat.)

where

$$H_{\theta\phi}(x) = \frac{n_1}{N} G_\theta(x) + \frac{n_2}{N} G_\phi(x) \quad .$$

Of course the conclusion is not valid if $\sigma^2_{\theta\phi}(B_N) = 0$. In particular, under the null hypothesis $\sigma^2_{\phi_0\phi_0} \equiv \sigma^2_{\phi_0}$ is given by

$$\frac{n_1 N}{n_2} \sigma^2_{\phi_0}(B_N) = \int_0^1 J_L^2(x)dx - (\int_0^1 J_L(x)dx)^2 = 1 .$$

In the non-null case we calculate

$$N\sigma^2_{\theta\phi}(B_N) = \frac{2n_2}{N} \left\{ \iint\limits_{-\infty<x<y<\infty} \frac{G_\phi(x)(1-G_\phi(y))g_\theta(x)g_\theta(y)}{H'_{\theta\phi}(x)H'_{\theta\phi}(y)} dJ_L(H(x))dJ_L(H(y)) \right.$$

$$\left. + \frac{n_2}{n_1} \iint\limits_{-\infty<x<y<\infty} \frac{G_\theta(x)(1-G_\theta(y))g_\phi(x)g_\phi(y)}{H'_{\theta\phi}(x)H'_{\theta\phi}(y)} dJ_L(H(x))dJ_L(H(y)) \right\}$$

$$= 4\frac{n_2}{N} \left\{ \frac{G_\phi(x^*)(1-G_\phi(x^*))g_\theta^2(x^*)}{(H'_{\theta\phi}(x^*))^2} + \frac{n_2}{n_1} \frac{G_\theta(x^*)(1-G_\theta(x^*))g_\phi^2(x^*)}{(H'_{\theta\phi}(x^*))^2} \right\} .$$

Of course this calculation restricts G to only those c.d.f.'s such that $H'_{\theta\phi}(x^*)$ is not zero where $\phi < x^* < \theta$.

4. Asymptotic Relative Efficiency

We now use the Pittman [8][7] criterion for the A.R.E. of two sequences of tests $N = \{W_N\}$ and $W^* = \{W_N^*\}$. Consider the sequence of alternatives $\triangle_N = \theta_N - \phi_N = kN^{-\frac{1}{2}}$ where k is a non-zero constant $\frac{\phi_N - \phi_0}{\theta_N - \theta_0} = -\frac{n_1}{n_2}$ and $\triangle = \theta - \phi$. Let $\lim \frac{n_1}{n_2} =$ $= \lim \frac{n_1^*}{n_2^*} = r$. Suppose W and W^* have the same size. The A.R.E. of W to W^* is defined to be $\lim\limits_{N\to\infty} \frac{N^*}{N}$ where N^* is the sample size of the second test required to achieve the same power for a

given alternative which W_N achieves with respect to the same alternatives when using a sample of N observations.

Consider such a sequence of alternatives with $\triangle = \theta - \phi$ and a sequence of statistics $W = \{W_N\}$ and $W^* = \{W_N^*\}$ satisfying the following three conditions in the neighborhood of $\triangle = 0(\phi=\theta==\phi_0)$:

$$\text{(a)} \quad \lim_{N\to\infty} P_G\left(\frac{W_N - E_{\theta\phi}(W_N)}{\sigma_{\theta\phi}(W_N)}\right) = \int_{-\infty}^{t} e^{-\frac{1}{2}x^2}\,dx$$

$$\text{(b)} \quad \lim_{N\to\infty} \frac{\sigma_{\theta_N\phi_N}(W_N)}{\sigma_{\phi_0}(W_N)} = 1$$

$$\text{(c)} \quad E_W(F) = \lim_{N\to\infty} \left\{ \frac{\frac{\partial}{\partial\triangle} E_{\theta\phi}(W_N)}{\left(\frac{n_1 n_2}{N}\right)^{\frac{1}{2}} \sigma_{\phi_0}(W_N)} \right\}^2_{\substack{\theta=\phi=\phi_0 \\ \triangle = 0}}$$

exists and is independent of k.

$E_N(F)$ is called by Pittman the efficacy of W at F and F refers to the distribution under which expectations are taken.

Pittman has shown that if (a), (b) and (c) are satisfied for $W = \{W_N\}$ and $W^* = \{W_N^*\}$ then

$$\lim_{N\to\infty} \frac{N^*}{N} = \frac{E_W(F)}{E_{W^*}(F)} = E_{W,W^*}(F)$$

(where $E_{W,W^*}(F)$ is the A.R.E. OF W to W^*), if $E_{W^*}(F) \neq 0$ and

$$\lim_{N\to\infty} \frac{n_1}{n_2} = \lim \frac{n_1^*}{n_2^*} = r .$$

To find $E_{B_N}(G)$ we need

$$\frac{\partial}{\partial\triangle} E_{\theta\phi}(B_N)\Big|_{\triangle=0} = \frac{\partial}{\partial\triangle}\left\{1 - 2G_\theta(x^*)\right\}\Big|_{\triangle=0}$$

$$= -\frac{2n_2}{N}\;\frac{\partial}{\partial\mu} G_\mu(\tilde{x})\Big|_{\mu=\phi_0}$$

where $\tilde{x}$ is the median of $g_{\phi_0}(x)$. To see this recall

$$H_{\theta\phi}(x) = \frac{n_1}{N} G_\theta(x) + \frac{n_2}{N} G_\phi(x)$$

$$= G_\theta(x) - \frac{n_2}{N}\triangle\;\frac{\partial}{\partial\mu} G_\mu(x)\Big|_{u=\hat{\phi}}$$

for some $\hat{\phi}$, $\theta \geq \hat{\phi} \geq \phi$.

Thus

$$2H_{\theta\phi}(x) - 2G_\theta(x) = -\frac{2n_2}{N}\triangle\frac{\partial}{\partial\mu}\;G_\mu(x)\Big|_{u=\hat{\phi}}$$

or

$$1 - 2G_\theta(x^*) = -\frac{2n_2}{N}\triangle\;\frac{\partial}{\partial\mu} G_\mu(x^*)\Big|_{\mu=\hat{\phi}} \qquad .$$

Differentiation yields

$$\frac{\partial}{\partial\triangle} E_{\theta\phi}(B_N)\Big|_{\substack{\triangle=0\\ \theta=\phi=\phi_0}} = -\frac{2n_2}{N}\;\frac{\partial}{\partial\mu}\;G_\mu(\tilde{x})\Big|_{\mu=\hat{\phi}_0}$$

as was indicated. We have then

$$E_B(G) = 4\Big[\frac{\partial}{\partial\mu}\;G_\mu(\tilde{x})\Big|_{\phi_0}\Big]^2 \quad .$$

Chernoff and Savage have pointed out that the asymptotic

efficiency of a sequence of tests can be established by means of comparison with a likelihood ratio test. They have also shown asymptotically that for C_N, compared to the t test, C_N is always more efficient except at the normal distribution[*] where the A.R.E. is unity. Capon has shown that under suitable regularity conditions, not satisfied by the double exponential distribution, the A.R.E. of the L.M.P.R.T. to the likelihood ratio test is unity. A natural question one may ask is whether the Chernoff-Savage result holds for all other distributions. A negative answer is obtained by investigating the double exponential case. That is, we shall show that the A.R.E. of Birnbaum's test to the likelihood ratio test L_N of the double exponential ranges from 0 to ∞ and is unity for L. We find

$$\begin{aligned} L_N &= \frac{1}{n_1}\left\{\text{number of } x_i > \phi_0 - \text{number of } x_i < \phi_0\right\} \\ &\quad - \frac{1}{n_2}\left\{\text{number of } y_i > \phi_0 - \text{number of } y_i < \phi_0\right\} \\ &= 1 - \frac{2}{n_1}\left\{\text{number of } x_i > \phi_0\right\} - 1 + \frac{2}{n_2} \\ &\qquad \left\{\text{number of } y_i > \phi_0\right\} \\ &= 2[L_y(\phi_0) - L_x(\phi_0)] \end{aligned}$$

where $L_\xi(\cdot)$ is the sample cumulative distribution function of a sample of n random observations on ξ. To calculate the efficiency of L_N at G we find

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[*] Chernoff and Savage considered distributions satisfying certain regularity conditions and Hodges and Lehmann [6] extended to arbitrary distributions.

$$E_{\theta\phi}(L_N|G) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta} \log L'(x,\theta)\Big|_{\theta=\phi_0} (g_\theta(x) - g_\phi(x))dx$$

$$= (\theta-\phi) \int_{-\infty}^{\infty} \frac{\partial}{\partial\theta} \log L'(x,\theta)\Big|_{\phi_0} \frac{\partial}{\partial\mu} g_\mu(x)\Big|_{\mu=\tilde{\phi}} dx$$

for some $\tilde{\phi}$, $\theta < \tilde{\phi} < \phi$.

Thus assuming g satisfies regularity conditions sufficient to allow interchange of differentiation and integration yields

$$\frac{\partial}{\partial\Delta} E_{\theta\phi}(L_N|G)\Big|_{\Delta=0} = -\int_{-\infty}^{\phi_0} \frac{\partial}{\partial\mu} g_\mu(x)\Big|_{\phi_0} dx + \int_{\phi_0}^{\infty} \frac{\partial}{\partial\mu} g_\mu(x)\Big|_{\phi_0} dx$$

$$= -\frac{\partial}{\partial\mu} G_\mu(x)\Big|_{\phi_0} \Big|_{-\infty}^{\phi_0} + \frac{\partial}{\partial\mu} G_\mu(x)\Big|_{\phi_0} \Big|_{\phi_0}^{\infty}$$

$$= -\frac{2\partial}{\partial\mu} G_\mu(\phi_0)\Big|_{\phi_0} + \frac{\partial}{\partial\mu} G_\mu(-\infty)_{\phi_0} + \frac{\partial}{\partial\mu} G_\mu(\infty)_{\phi_0} \quad .$$

Also

$$\sigma^2_{\phi_0}(L_N|G) = \left(\frac{1}{n_1} + \frac{1}{n_2}\right) .$$

A particularly simple case results when we consider translation or scale alternatives, i.e. $G_\theta(x)$ of the form $G(x-\theta)$ or $G(\frac{x}{\theta})$. Then combining our results yields

$$E_{L_N,B_N}(G) = \frac{g^2_{\phi_0}(\phi_0)}{g^2_{\phi_0}(\tilde{x})} \quad .$$

Clearly when G is the double exponential the A.R.E. is unity as it is whenever ϕ_0 and the median of g coincide. However g may

be chosen so that $E_{L_N,B_N}(G)$ ranges between zero and infinity. For example, if

$$g_\theta(x) = \frac{1}{\theta x\sqrt{2\pi}}\, e^{-\lg^2 \frac{(x/\beta)}{2\theta^2}}$$

then the median $\tilde{x} = \beta$. Thus, at $\phi_0 = 1$,

$$E_{L_N,B_N}(G) = \left[\beta e^{-\frac{1}{2}\lg^2(\beta)}\right]$$

and as β gets sufficiently close to 0, the A.R.E. becomes arbitrarily small.

Similarly, let

$$g_\theta(x) = \begin{cases} \dfrac{(a\theta+\theta^2 x)}{2(a+1)}\, e^{-\theta x} & x > 0 \\ \dfrac{(a\theta-\theta^2 x)}{2(a+1)}\, e^{\theta x} & x < 0 \end{cases}$$

where $a \geq 0$ and $\theta > 0$. Then the median $\tilde{x} = 0$ and at $\phi_0 = 1$

$$E_{L_N,B_N}(G) = \frac{(a+1)e^{-1}}{a} \quad .$$

As a gets sufficiently close to 0, the A.R.E. becomes arbitrarily large.

5. A remark on the relationship between B_N and Mood's M test

It is interesting to compare the Birnbaum test and Mood's test M. [3] For the latter; asymptotic normality has been established [3] with mean $n_1 G_\theta(x^*)$ and variance

$$\left[\frac{n_2}{n_1 n_2 F_\theta(x^*)(1-F_\theta(x^*))} + \frac{n_1}{n_1 n_2 F_\phi(x^*)(1-F_\phi(x^*))}\right]^{-1} .$$

Calculation similar to those for B_N yield

$$E_M(G) = 4 \left[\frac{\partial}{\partial\mu} G_\mu(\tilde{x}) \Big|_{\mu=\phi_0} \right]^2$$

Thus M and B_N have A.R.E. unity for all distributions satisfying the above regularity conditions. In particular, this gives an asymptotic justification for use of Mood's test: When the double exponential shape is true, Mood's test has A.R.E. unity (B_N seems preferable under this assumption since it has the exact locally best property as well as A.R.E. unity).

6. Acknowledgement

The author is indebted to Professor Allan Birnbaum for suggesting the problem and for many helpful discussions.

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